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Regional Pole Placement Method for Discrete-Time Systems

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I. Introduction

THE dynamic characteristics of a linear system are influenced by the location of its poles. In general, it is sufficient that poles be placed in a suitable region of the complex plane, instead of placing them at their respective exact positions. The regional pole placement (RPP) technique was studied for continuous-time systems for placing the eigenvalues to the left of sector for a well-damped response.[1] The problem of designing feedback gains to optimally place the closed-loop poles of a discrete-time system has been investigated using a linear quadratic regulator (LQR) to obtain a linear state feedback law guaranteeing that the closed-loop poles lie inside a circle with given radius and centered at a distance on the real axis.[2,3] However, there appears to be no attempt made so far for the problem of placing the eigenvalues within a logarithmic spiral in the z plane for discrete-time systems.

In this Engineering Note, for discrete-time systems, a novel method of placing the eigenvalues corresponding to the region within a logarithmic spiral (which actually translates into a well-damped continuous-time system) is developed. Essentially, the method proposes a simple technique of fitting a circle within a logarithmic spiral.

II. LQR Theory for RPP in Discrete-Time Systems

Consider a linear, time-invariant, discrete-time controllable system

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$

where x is an $n \times 1$ state vector, u is an $r \times 1$ control vector, and A and B are $n \times n$ and $n \times r$ constant matrices, respectively. We formulate an optimal control problem such that the optimal control minimizes a specified performance index while at the same time places the closed-loop poles inside a circular region. Given the plant dynamics of Eq. (1) and the performance index

$$J = \sum_{k=0}^{\infty} \left(\frac{1}{\alpha}\right)^{2k} [x^T(k)Qx(k) + u^T(k)Ru(k)] \tag{2}$$

where $Q = Q^T \geq 0$ and $R = R^T > 0$, it is well known[3] that the optimal control that minimizes Eq. (2) has closed-loop poles inside a circle centered at the origin with radius $r = \alpha$. Now, if

$$\bar{x}(k+1) = A_\beta \bar{x}(k) + B\bar{u}(k), \qquad A_\beta = A - \beta I \tag{3}$$

then the optimal control that minimizes

$$J = \sum_{k=0}^{\infty} \left(\frac{1}{\alpha}\right)^{2k} [\bar{x}^T(k)Q\bar{x}(k) + \bar{u}(k)^T R\bar{u}(k)] \tag{4}$$

has all of the closed-loop poles inside the region of the circle (of radius α and center β), as shown in Fig. 1. Thus, the problem reduces to finding a performance index so that the resultant optimal control subject to Eq. (1) is equivalent to the optimal control that minimizes Eq. (4) subject to Eq. (3). This problem can be further reduced to a standard LQR problem. Indeed, if we let

$$\hat{x}(k) = (1/\alpha)^k \bar{x}(k), \qquad \hat{u}(k) = (1/\alpha)^k \bar{u}(k) \tag{5}$$

$$\hat{A} = (1/\alpha)A_\beta, \qquad \hat{B} = (1/\alpha)B \tag{6}$$

then the dynamical equation (3) becomes

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}\hat{u}(k) \tag{7}$$

and the performance index of Eq. (4) can be written as

$$J = \sum_{k=0}^{\infty} [\hat{x}(k)Q\hat{x}(k) + \hat{u}(k)R\hat{u}(k)] \tag{8}$$

The minimization problem with plant dynamics of Eq. (1) and the performance index of Eq. (2) is reduced to a standard LQR problem with plant dynamics of Eq. (7) and the performance index of Eq. (8). The optimal control law that minimizes Eq. (8) subject to the constraint of Eq. (11) is

$$\hat{u}(k) = -F\hat{x}(k) \tag{9}$$

where

$$F = [R + \hat{B}^T P \hat{B}]^{-1} \hat{B}^T P \hat{A} \tag{10}$$

and P is the symmetric, positive-definite solution of the algebraic Riccati equation

$$P = Q + \hat{A}^T P \hat{A} - \hat{A}^T P \hat{B}[R + \hat{B}^T P \hat{B}]^{-1} B^T P \hat{A} \tag{11}$$

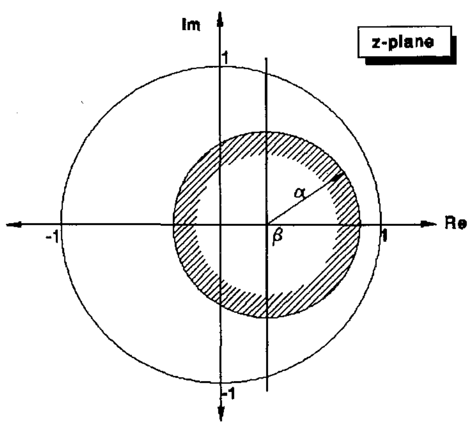

Fig. 1 Regional pole placement in discrete-time system.

Received Oct. 30, 1995; revision received April 2, 1996; accepted for publication April 4, 1996.

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Moreover, if $[\hat{A}, \hat{B}]$ is either completely controllable or stabilizable and if $[\hat{A}, D]$ is completely observable, where D is any $n \times n$ matrix such that $DD^T = Q$, then the closed-loop feedback system

$$\hat{x}(k+1) = (\hat{A} - \hat{B}F)\hat{x}(k) \tag{12}$$

is asymptotically stable. Since the eigenvalues of the stable closed-loop system $(\hat{A} - \hat{B}F)$ are less than 1, it follows that the eigenvalues of $(A_\beta - BF)$ are less than α. Hence, the eigenvalues of

$$\boldsymbol{x}(k+1) = (A - BF)\boldsymbol{x}(k) \tag{13}$$

are all inside a circle centered at $(\beta, 0)$ with radius α. Thus, the optimal control that minimizes Eq. (2) subject to plant dynamics Eq. (1) ensures that all of the closed poles are inside a circle centered at $(\beta, 0)$ with radius α.

III. Fitting a Circle Within a Logarithmic Spiral

If a certain maximum damping ratio is specified in the design, all of the characteristic equation roots must lie to the left of the constant damping ratio in the s plane or inside the constant damping ratio logarithmic spiral in the z plane. Then, we fit a circle that has maximum area within the logarithmic spiral. The logarithmic spiral is described by[2]

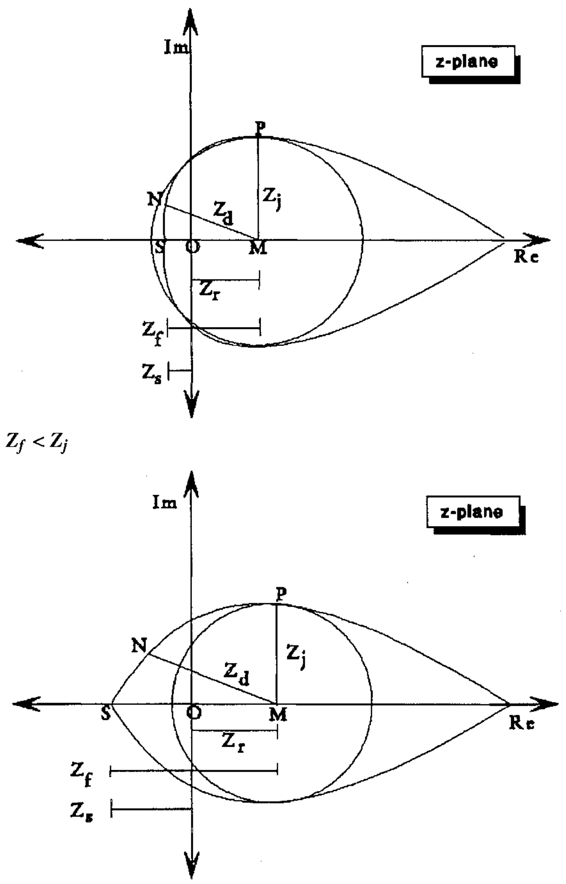
$$z = e^{\sigma T} e^{j\omega T} = e^{\sigma T} \angle \omega T, \qquad \omega = \sigma \tan\gamma \tag{14}$$

Then, relation (14) can be written as

$$z = e^{\sigma T}[\cos(\sigma T \tan\gamma) + j \sin(\sigma T \tan\gamma)] \tag{15}$$

The maximum radius for the circle is determined by first finding the maximum imaginary part of Eq. (15). Thus, by differentiating the imaginary part of Eq. (15) with respect to σ and equating it to zero, we get

$$\tan(\sigma_m T \tan\gamma) = -\tan\gamma \tag{16}$$



a) $Z_f < Z_j$

b) $Z_f > Z_j$

Fig. 2 Fitting a circle inside a spiral.

From Eqs. (15) and (16), we have $M = e^{\sigma_m T}\cos(\sigma_m T \tan\gamma)$ and $P = e^{\sigma_m T}\sin(\sigma_m T \tan\gamma)$, where M and P are the x and y coordinates (in the z plane) of the circle with maximum radius, as shown in Figs. 2a and 2b. Also, the following relationships are easily established:

$$Z_r = \left[e^{\sigma_m T}\cos(\sigma_m T \tan\gamma)\right] \tag{17a}$$

$$Z_s = \left[e^{\sigma_1 T}\cos(\sigma_1 T \tan\gamma)\right] \tag{17b}$$

$$Z_f = \left[e^{\sigma_m T}\cos(\sigma_m T \tan\gamma) - e^{\sigma_1 T}\cos(\sigma_1 T \tan\gamma)\right] \tag{17c}$$

$$Z_j = \left[e^{\sigma_m T}\sin(\sigma_m T \tan\gamma)\right] \tag{17d}$$

$$Z_d^2 = \left[e^{\sigma T}\cos(\sigma T \tan\gamma) - e^{\sigma_m T}\cos(\sigma_m T \tan\gamma)\right]^2 + [e^{\sigma T}\sin(\sigma T \tan\gamma)]^2 \tag{17e}$$

where σ_1 and σ_m correspond to points S and P, respectively, on the spiral and σ corresponds to any point N on the spiral.

IV. Algorithm for RPP

We now give the various steps involved in the procedure for placing the eigenvalues within the region specified by a circle inside the spiral.

Step 1) For a given γ corresponding to a constant damping ratio and sampling interval T, Eq. (16) gives the corresponding σ_m. Compute Z_r, Z_s, Z_s, Z_f, and Z_d from Eq. (17). Now there are two cases: Case 1, when $Z_f < Z_j$, as shown in Fig. 2a, we have $Z_f < Z_d$. Then the circle with radius MP overlaps the spiral and hence cannot be used for pole placement. Instead we take a circle with radius $\alpha = Z_f$, and center at $\beta = OM$. Case 2, when $Z_f > Z_j$, we have $Z_j < Z_d$. Then the circle with radius MP will not overlap the spiral; therefore, we take a circle with radius $\alpha = Z_j$ and center at $\beta = OM$.

Step 2) For a given performance index of Eq. (2), let us assign the matrices Q and R.

Step 3) For values of α and β evaluated in step 1, obtain A_β, $\hat{A}$, and $\hat{B}$ from Eqs. (3) and (6).

Step 4) Solve Eq. (11) and obtain the matrix P and the resultant feedback gain F. The resultant closed-loop system is $(A - BF)$.

Let us illustrate the enumerated method by an example. Given a discrete-time system as

$$x(k+1) = \begin{bmatrix} 0.1 & 0.3 & 1 \\ 2.3 & 1 & 0 \\ 0.1670 & 0.23 & 0.8 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(k)$$

whose eigenvalues are = {−0.2014, 0.3039, 1.7975}, it is desired to place these eigenvalues inside a spiral corresponding to $\gamma = 135$ deg. Using the preceding algorithm, we finally find $\alpha = 0.32$ and $\beta = 0.32$ and $F = [-65.2032 \;\; 4.2517 \;\; 66.3158]$. The eigenvalues of the closed-loop system $(A - BF)$ are = {0.1232, 0.3186, 0.3457}, which are located inside the specified region of the circle.

The method of placing the eigenvalues within two circles for a two time-scale discrete-time system has been reported elsewhere.[4]

V. Conclusions

Using LQR theory, an RPP method for discrete-time systems has been presented. The method developed for the RPP embeds a circle inside a logarithmic spiral in the complex z plane.

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Dynamic Boundary Evaluation Method for Approximate Optimal Lunar Trajectories

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Introduction

RESEARCH into low-thrust Earth–moon trajectories has increased recently.[1–3] The time and effort required to obtain an optimal low-thrust lunar trajectory depends greatly on the initial estimate of the trajectory used in the optimizing process. Pierson and Kluever[2] present an effective method for finding an initial estimate for a class of Earth–moon trajectories. However, their method requires the setup of tabular data before the initial estimate can be found. This data table must be recomputed for each new initial thrust-to-weight ratio, radius of the initial low Earth orbit (LEO), or radius of the final low lunar orbit (LLO). This Note describes a dynamic boundary evaluation (DBE) method that replaces the data table construction and subsequent interpolation with an on-line iterative numerical process.

The optimal low-thrust lunar trajectory problem treated here involves a planar transfer from a circular LEO to a circular LLO. A specified thrust-coast-thrust sequence is assumed using constant thrust. The spacecraft thrusts away from the LEO, in an increasing spiral, until enough energy is gained to make the lunar rendezvous. The spacecraft then coasts to the vicinity of the lunar sphere of influence, where the spacecraft thrusts, in a decreasing spiral, to the desired LLO. The minimum-fuel solution is the one that accomplishes this transfer with the minimum time spent in the two thrusting periods, because the assumed constant thrust implies a constant mass flow. The Pierson–Kluever method[2] uses maximum-energy spirals, described later, to approximate both the Earth-departure and the lunar-capture thrusting arcs. This composite trajectory, consisting of the maximum-energy spirals plus an optimal translunar coast, is then used to estimate the fully minimum-fuel transfer from a circular LEO to a circular LLO. This estimate is very accurate in terms of performance; i.e., the fuel spent is very close to that for the fully optimal case, and the control time histories for the thrust-steering angle are adequate to ensure convergence to a solution of the full problem.

Implementation of the DBE Method

The approximate optimal lunar trajectory consists of a nonthrusting arc that patches two thrusting arcs together. The two thrusting portions of the trajectory include the departure from LEO and the capture into LLO. These two thrusting phases are modeled by maximum-energy spirals, and patching the trajectories then matches positions and velocities at both the end of the Earth departure and the beginning of the lunar capture. The problem statement for the coast-patch problem can be written as follows: Find r_1 (the radius at the beginning of the coast phase), θ_1 (the polar angle at the beginning of the coast phase), m_f (the final mass), and a nondimensional time parameter $\alpha (t = \alpha\tau)$ that minimize

$$J = t_{dep} + t_{cap} \tag{1}$$

Received July 10, 1995; revision received Feb. 19, 1996; accepted for publication March 11, 1996.

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subject to restricted three-body dynamics[4] with no thrusting,

$$\frac{dx}{d\tau} = \alpha f(x), \qquad 0 \le \tau \le 1 \tag{2}$$

and the boundary conditions

$$r(0) = r_1 \tag{3}$$

$$\theta(0) = \theta_1 \tag{4}$$

$$V_r(0) = f_1(r_1) \tag{5}$$

$$V_\theta(0) = f_2(r_1) \tag{6}$$

$$V_r(1) = g_1(r_2, m_f) \tag{7}$$

$$V_\theta(1) = g_2(r_2, m_f) \tag{8}$$

where $(r, \theta, V_r, V_\theta)$ are the radius, polar angle, and radial and tangential velocities of a standard polar coordinate system, respectively; r_2 is the radius at the beginning of the lunar capture; and t_{dep} and t_{cap} are the departure and capture time durations, respectively.

$$t_{dep} = t_{dep}(r_1) \tag{9}$$

$$t_{cap} = t_{cap}(r_2, m_f) \tag{10}$$

Use of the time transformation $t = \alpha\tau$ allows us to convert a variable end-time problem in t into a fixed end-time problem in τ. The functions f_1 and f_2 provide the boundary conditions at the end of the Earth departure and require one input parameter, the radius r_1, whereas the functions g_1 and g_2 provide the boundary conditions at the beginning of the lunar capture and require two input parameters, the radius r_2 and the final mass m_f. These functions can be determined in different ways, one of which is to construct, from a set of maximum-energy solutions of fixed duration, a table of time-of-flight and radial and tangential velocity components for the specific initial thrust-to-weight ratio used and then simply to interpolate within this table to find the boundary conditions[2] for the translunar coast trajectory.

On-Line Approach

We first tried to solve these boundary conditions on-line using a root-finding process.[5] The maximum-energy problem mentioned earlier is for a fixed final time and free final radius. The related problem of fixing the final radius and finding the velocity that maximizes the final total energy is actually a maximum final-speed problem. The use of maximum final-speed rather than maximum final-energy trajectories for the escape/capture approximations is possible but beyond the scope of this Note. The problem of finding the maximum-energy spiral that ends at a specified radius is equivalent to finding the time-of-flight t_{dep} that satisfies

$$r_{desired} - r(t_{dep}) = 0 = F(t_{dep}) \tag{11}$$

The expression $r(t_{dep})$ denotes the final radius found by solving a maximum-energy problem with a time-of-flight of t_{dep}. Equation (11) requires evaluating a function, however complicated, that is the difference between the desired final radius and the calculated final radius. The departure time that satisfies this equation completely determines that trajectory, and the result can be used as the initial conditions for the coast-patch problem, Eqs. (1–8). An iterative root-finding algorithm then can be used to solve Eq. (11). This method proved effective, but its performance was limited by the necessary computational burden of calculating a large number of maximum-energy spiral solutions for each iteration of the optimization. We then derived a new method of finding the boundary conditions by incorporating the root-finding process into the optimal solution of the approximate translunar problem.

A still more direct approach is to treat the radius r as the independent variable rather than time. Then, the final radius is automatically the desired one and we have a fixed end-time problem. This conversion is possible because the radius is a nondecreasing function